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## AN EQUIVARIANT TRANSVERSALITY THEOREM AND ITS APPLICATIONS

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*To the memory of the late Professor Minoru Nakaoka*

**Abstract.** Let  $G$  be a finite group. In this article, we recall an equivariant transversality theorem and discuss its applications to semifree  $G$ -actions on closed manifolds and to Smith-equivalent real  $G$ -modules.

### 1. INTRODUCTION

Unless otherwise stated, let  $G$  be a finite group. We mean by a *manifold* a paracompact smooth manifold. A *submanifold*,  $M$  say, of a manifold,  $N$  say, should be read as a regular smooth submanifold such that  $M$  is a closed subset of  $N$ . We mean by a  $G$ -*manifold* a smooth manifold with a smooth  $G$ -action. In particular, each connected component of a manifold in the present article is  $\sigma$ -compact, and an arbitrary  $G$ -manifold can be equipped with a  $G$ -invariant Riemannian metric.

Let  $M$  and  $N$  be manifolds,  $B$  a subset of  $M$ ,  $Y$  a submanifold of  $N$ , and  $f : M \rightarrow N$  a continuous map. We say that  $f$  is *transversal on  $B$  to  $Y$  in  $N$*  if  $f$  is smooth on a neighborhood of  $f^{-1}(Y) \cap B$  in  $M$  and the linear map

$$T_x(M) \xrightarrow{df_x} T_y(N) \longrightarrow T_y(N)/T_y(Y)$$

is surjective for every  $y \in Y$  and  $x \in f^{-1}(y) \cap B$ , where  $T_x(M)$  stands for the tangent space of  $M$  at  $x$ . There have been obtained several versions of equivariant transversality theorems, e.g. A. Wasserman [19, Lemma 3.3], T. Petrie [16, §1,

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p.188], E. Bierstone [1, Theorem 1.3]. In this paper we will discuss applications of the next version.

**Theorem 1.1.** *Let  $M$  be a  $G$ -manifold,  $N$  a  $G$ -manifold with a  $G$ -invariant Riemannian metric,  $A$  a  $G$ -invariant closed subset of  $M$ , and  $Y$  a  $G$ -submanifold of  $N$ . Let  $f : M \rightarrow N$  be a smooth  $G$ -map transversal on  $A$  to  $Y$  in  $N$ . Suppose the  $G$ -action on  $M \setminus A$  is free. Then for an arbitrary  $G$ -invariant positive continuous function  $\delta : M \rightarrow \mathbb{R}$ , there exists a smooth  $G$ -map  $g : M \rightarrow N$  satisfying the following conditions.*

- (1)  $g$  is transversal on  $M$  to  $Y$  in  $N$ .
- (2)  $g|_A = f|_A$ .
- (3)  $d_N(f(x), g(x)) < \delta(x)$  for all  $x \in M$ , where  $d_N$  stands for the distance function on  $N$  induced from the Riemannian metric of  $N$ .

We mean by a real (resp. complex)  $G$ -module a real (resp. complex)  $G$ -representation space of finite dimension. For a real  $G$ -module  $V$  (of finite dimension), let  $S(V)$  denote the unit sphere of  $V$  with respect to some  $G$ -invariant inner product on  $V$ . The following two propositions have been known.

**Proposition** ([14, Lemma 2.1]). *If  $G$  is a group of order 2 and  $M$  is a connected closed  $G$ -manifold of positive dimension then  $|M^G| \neq 1$ .*

**Proposition** ([7, Lemma 2.2]). *If  $M$  is a connected closed orientable  $G$ -manifold of positive dimension such that the  $G$ -action on  $M \setminus M^G$  is free, then  $|M^G| \neq 1$ .*

The latter proposition is generalized to the next result.

**Theorem 1.2.** *Let  $M$  be a connected closed oriented  $G$ -manifold of dimension  $n+1$ , and  $\Sigma$  an oriented homotopy sphere of dimension  $n$ . Suppose the  $G$ -action on  $M$  is semifree and preserves the orientation of  $M$ . If  $M^G$  is a finite set then the congruence*

$$\sum_{x \in M^G} \deg(f_x) \equiv 0 \pmod{|G|}$$

*holds, where  $T_x(M)$  is the tangent space of  $M$  at  $x$  and  $f_x$  is an arbitrary (continuous)  $G$ -map  $S(T_x(X)) \rightarrow \Sigma$  for each  $x \in M^G$ .*

**Theorem 1.3.** *Let  $M$  be a connected closed oriented  $G$ -manifold of positive dimension such that the  $G$ -action on  $M$  is semifree and  $M^G = \{a, b\}$ . Then the spheres  $S(T_a(M))$  and  $S(T_b(M))$  are  $G$ -homotopy equivalent to each other.*

**Corollary 1.4.** *Let  $V$  and  $W$  be real  $G$ -representations satisfying  $\dim V = \dim W$ . If  $S(V) \amalg S(W)$  is the boundary of a compact orientable  $G$ -manifold  $M$  such that the  $G$ -action on  $M$  is free then  $S(V)$  and  $S(W)$  are  $G$ -homotopy equivalent to each other.*

A homotopy sphere  $\Sigma$  with a  $G$ -action is called a *Smith sphere* if  $\Sigma^G$  consists of exactly 2 points. Two real  $G$ -modules  $V$  and  $W$  are said to be *Smith equivalent* if there exists a Smith sphere  $\Sigma$  such that  $\Sigma^G = \{a, b\}$  with  $V \cong T_a(\Sigma)$  and  $W \cong T_b(\Sigma)$  as real  $G$ -modules.

**Theorem 1.5.** *Let  $G$  be a finite group and let  $V$  and  $W$  be Smith-equivalent real  $G$ -modules. For any normal subgroup  $H$  of  $G$  such that  $|G/H|$  is a prime and a Sylow 2-subgroup  $H_2$  of  $H$  is normal in  $H$ ,  $S(V^H)$  and  $S(W^H)$  are  $G/H$ -homotopy equivalent to each other.*

## 2. TRANSVERSALITY OF MAPS

In this section, let us recall classical transversality theorems. First, recall a result by A. Wasserman.

**Lemma** (A. Wasserman [19, Lemma 3.3]). *Let  $G$  be a compact Lie group,  $M$  and  $N$   $G$ -manifolds,  $f : M \rightarrow N$  a smooth  $G$ -map,  $W \subset N$  a closed invariant submanifold, and  $C$  a closed subset of  $M^G$ . Suppose  $f|_{M^G}$  is transversal on  $C$  to  $W^G$  in  $N^G$ . Then there exists a homotopy  $f_t$  such that  $f_0 = f$ ,  $f_t|_C = f|_C$ , and  $f_1|_{M^G}$  is transversal on  $M^G$  to  $W^G$  in  $N^G$ .*

T. Petrie gave several versions and the next one may be most basic.

**Proposition** (Petrie [16, §1, p.188]). *Let  $G$  be a compact Lie group. Let  $M$ ,  $N$  and  $Y \subset N$  be  $G$ -manifolds and  $f : M \rightarrow N$  a proper  $G$ -map. Suppose  $f : M \rightarrow N$  is transversal to  $Y$  on a  $G$ -neighborhood of a closed subset  $Z$  of  $M$ . Let  $K$  be a maximal closed subgroup such that  $(M \setminus Z)^K \neq \emptyset$ . Then  $f$  is  $G$ -homotopic rel  $Z$  to a proper  $G$ -map  $h : M \rightarrow N$  such that  $h^K$  is transversal on  $M^K$  to  $Y^K$  in  $Z^K$ .*

As its proof, T. Petrie wrote as follows. (Note that  $N(K)/K$  acts freely on  $M^K \setminus Z$ .) “This uses the Thom Transversality Lemma [11] for the case of trivial group action and the  $G$ -homotopy extension lemma [19, Lemma 3.2].” Here the reference [11] should be replaced by an appropriate one.

Another version is obtained by E. Bierstone’s theory, namely from the following three results.

**Theorem** (Bierstone [1, Theorem 1.3]). *Let  $G$  be a compact Lie group. If  $P$  is a closed  $G$ -submanifold of  $N$ , then the set of smooth equivariant maps  $F : M \rightarrow N$  which are in general position with respect to  $P$  is open in Whitney topology.*

**Theorem** ([1, Theorem 1.4]). *Let  $G$  be a compact Lie group. If  $P$  is an invariant submanifold of  $N$ , then the set of smooth equivariant maps  $F : M \rightarrow N$  which are in general position with respect to  $P$  is a countable intersection of open dense sets (in the Whitney of  $C^\infty$  topology).*

**Proposition** ([1, Proposition 6.3]). *If a smooth equivariant map  $F : M \rightarrow N$  is in general position with respect to an invariant submanifold  $P$  of  $N$ , then it is stratumwise transversal to  $P$ . In other words, for every isotropy subgroup  $H$  of  $M$ ,  $F|_{M^H} : M^H \rightarrow N^H$  is transversal to  $P^H$ .*

Our Theorem 1.1 is an equivariant analogue of A. Hattori [6, Ch.6, §3, Theorem 3.6].

### 3. MAPS BETWEEN SPHERES

We mean by a *homotopy sphere* a closed manifold being homotopy equivalent to a sphere. Let  $X$  be a finite  $G$ -CW complex such that  $G$  acts freely on  $X$ . For a  $G$ -map  $f : X \rightarrow X$ , the Lefschetz number  $L(f)$  is congruent to 0 mod  $|G|$ . In the case where  $X$  is a homotopy sphere of dimension  $n$ , we have

$$(3.1) \quad L(f) = 1 + (-1)^n \deg f \equiv 0 \pmod{|G|}.$$

Using this property, we can prove the next fact without difficulties.

**Lemma 3.1** ([17, 4, 9]). *Let  $X$  be a connected homotopy sphere with a free  $G$ -action. Then for any  $G$ -map  $f : X \rightarrow X$ ,  $\deg f$  is congruent to 1 mod  $|G|$ .*

In addition, by standard arguments using Steenrod's obstruction theory [18], we can prove the next fact.

**Lemma 3.2** ([17, 4]). *Let  $X$  and  $Y$  be connected homotopy spheres of same dimension with free  $G$ -actions. Then the following conclusions hold.*

- (1) *There exist a  $G$ -map  $X \rightarrow Y$  and a  $G$ -map  $Y \rightarrow X$ .*
- (2) *For any  $G$ -maps  $f_0, f_1 : X \rightarrow Y$ ,  $\deg f_0 \equiv \deg f_1 \pmod{|G|}$ .*
- (3) *For any  $G$ -map  $f_0 : X \rightarrow Y$  and any integer  $m$ , there exists a  $G$ -map  $f_1 : X \rightarrow Y$  such that  $\deg f_1 = \deg f_0 + m|G|$ .*

These lemmas provide the next proposition.

**Proposition 3.3.** *Let  $X$  and  $Y$  be connected homotopy spheres of same dimension with free  $G$ -actions and let  $f : X \rightarrow Y$  be a  $G$ -map. Then  $\deg f$  is prime to  $|G|$ .*

*Proof.* By Lemma 3.2, there is a  $G$ -map  $g : Y \rightarrow X$ . Moreover by Lemma 3.1 we have

$$\deg(g \circ f) \equiv 1 \pmod{|G|}$$

and  $\deg(g \circ f) = \deg g \cdot \deg f$ . Thus  $\deg f$  is prime to  $|G|$ .  $\square$

#### 4. TANGENTIAL REPRESENTATIONS

Let  $V$  be a real  $G$ -module such that the  $G$ -action on  $V \setminus \{0\}$  is free. We adopt an orientation of the ambient space of  $V$ . Let  $\Sigma$  be an oriented homotopy sphere equipped with a free smooth  $G$ -action such that  $\dim \Sigma = \dim S(V)$ . Then there exists a smooth  $G$ -map  $f_{V,\Sigma} : S(V) \rightarrow \Sigma$  and  $\deg(f_{V,\Sigma})$  is prime to  $|G|$ .

*Proof of Theorem 1.2.* Let us fix an arbitrary point  $a \in M^G$ . The tangential representation  $V = T_a(M)$  has the orientation inherited from that of  $M$ . Since the  $G$ -action on  $S(V)$  preserves the orientation,  $\dim S(V)$  is odd. Without any loss of generality, we can assume  $\Sigma = S(V)$ . Set  $Y = S(\mathbb{R} \oplus V)$ . There is a canonical orientation preserving  $G$ -diffeomorphism from the  $G$ -disk  $D(V)$  to the upper hemisphere  $S_+$  of  $Y$ . This diffeomorphism carries the center of  $D(V)$  to the north pole  $p_+ = (1, 0)$  of  $Y$ , where  $1 \in \mathbb{R}$  and  $0 \in V$ . Take small  $G$ -disk neighborhoods  $D_x (\cong D(T_x(M)))$  of points  $x \in M^G$  in  $M$ , respectively, so that  $D_{x_1} \cap D_{x_2} = \emptyset$

for distinct  $x_1, x_2 \in M^G$ . For each point  $x \in M^G$ , there is a smooth  $G$ -map  $f_x : \partial D_x \rightarrow S(V) = \partial S_+$ . Let  $Df_x : D_x \rightarrow D(V) = S_+$  denote the radial extension of the map  $f_x$ . Clearly  $Df_x$  is transversal on a color neighborhood of  $\partial D_x$  to  $p_+$  in  $Y$ . In addition, it holds that

$$\deg(Df_x : (D_x, \partial D_x) \rightarrow (S_+, \partial S_+)) = \deg(f_x : \partial D_x \rightarrow \partial S_+).$$

Set  $X = M \setminus \coprod_{x \in M^G} \text{Int}(D_x)$ . Then the  $G$ -action on  $X$  is free. Since  $S_+$  is contractible, the  $G$ -map  $\coprod_{x \in M^G} Df_x$  extends to a continuous  $G$ -map  $f : M \rightarrow S_+$  such that  $f$  is smooth on  $X$ . We will regard  $f$  as a map  $M \rightarrow Y$  as well. For a  $G$ -invariant positive function  $\delta : M \rightarrow \mathbb{R}$ , take a  $G$ -equivariant  $\delta$ -approximation  $g : M \rightarrow Y$  of  $f$  such that

- (1)  $g$  is  $G$ -homotopic to  $f$  relatively to  $\coprod_{x \in M^G} D_x$ , and
- (2)  $g|_X$  is smooth and transversal on  $X$  to  $\{p_+\}$  in  $Y$ .

Since the  $G$ -action on  $g^{-1}(p_+) \cap X$  is free, each  $G$ -orbit in  $g^{-1}(p_+) \cap X$  consists of  $|G|$  points. Thus it holds that

$$\deg(g) \equiv \sum_{x \in M^G} \deg(f_x : \partial D_x = S(T_x(M)) \rightarrow S(V)) \pmod{|G|}.$$

On the other hand, the equality  $\deg(g) = \deg(f) = 0$  follows from the fact that  $f : M \rightarrow Y$  is not a surjection. Hence we can conclude

$$\sum_{x \in M^G} \deg(f_x) \equiv 0 \pmod{|G|}.$$

□

*Proof of Theorem 1.3.* Set  $V = T_a(M)$  and  $W = T_b(M)$ . Then  $G$  freely acts on  $S(V)$  and  $S(W)$ .

If  $|G| = 2$  then  $V$  and  $W$  are isomorphic as real  $G$ -representations, and hence  $S(V)$  and  $S(W)$  are  $G$ -diffeomorphic.

Thus we consider the other case, namely one where  $|G| \geq 3$ . The real  $G$ -modules  $V$  and  $W$  have the inherited orientations from that of  $M$ , respectively. Since the  $G$ -action on  $S(V)$  preserves the orientation, so does the  $G$ -action on  $M$ . Let  $f_{V,V}$  be the identity map on  $S(V)$  and take a smooth  $G$ -map  $f_{W,V} : S(W) \rightarrow S(V)$ . By

Theorem 1.2, we get

$$\deg(f_{V,V}) + \deg(f_{W,V}) = 1 + \deg(f_{W,V}) \equiv 0 \pmod{|G|},$$

and hence  $\deg(f_{W,V}) \equiv -1 \pmod{|G|}$ . Thus there is a smooth  $G$ -map  $f : S(W) \rightarrow S(V)$  satisfying  $\deg(f) = -1$ . On the other hand, there exists a smooth  $G$ -map  $h : S(V) \rightarrow S(W)$ . We have  $\deg(h \circ f) \equiv 1 \pmod{|G|}$  and hence  $\deg(h) \equiv -1 \pmod{|G|}$ . There exists a smooth  $G$ -map  $g : S(V) \rightarrow S(W)$  such that  $\deg(g) = -1$ . These  $f$  and  $g$  are  $G$ -homotopy inverses to each other.  $\square$

*Proof of Theorem 1.5.* Set  $\Sigma^G = \{a, b\}$ . If a connected component  $A$  of  $\Sigma^H$  containing either  $a$  or  $b$  has positive dimension then by Proposition 1  $A$  contains both  $a$  and  $b$ . By Theorem 1.3,  $S(V^H)$  and  $S(W^H)$  are  $G/H$ -homotopy equivalent. If  $\dim V^H = 0$  and  $\dim W^H = 0$  both hold then  $S(V^H)$  and  $S(W^H)$  are the empty set and hence they are  $G/H$ -homotopy equivalent.  $\square$

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